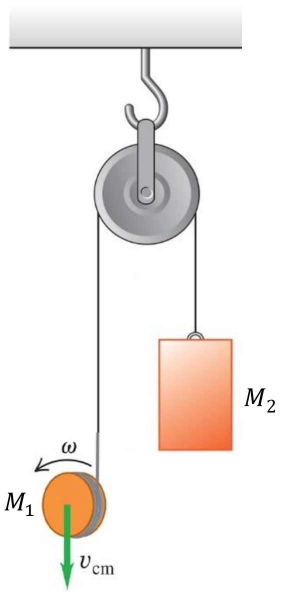
PHYS 101 – General Physics I Final Exam Solution

Duration: 150 minutes | Monday, 5 June 2023; 18:00

1. A disc of mass M_1 and radius R ($I = M_1R^2/2$) is wrapped with an ideal string and is connected to a mass M_2 over a massless and frictionless pulley as shown in the figure. When the system is released from rest, both the center of the disc and the mass M_2 start **moving down** with **the same acceleration**. Gravitational acceleration is given as g, and assume that the string tied to the disc stays vertical throughout the motion.

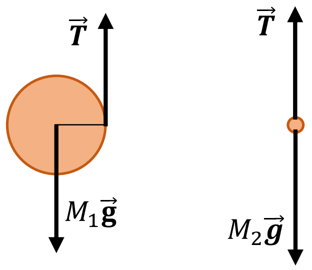


(a) (4 Pts.) Draw the free body diagram for both masses.

(b) (12 Pts.) What is the ratio M_1/M_2?

(c) (4 Pts.) What is the angular acceleration of the disc?

Solution: (a, b) Writing Newton's second law for the rotational motion of the disc, we have



$$RT = \frac{1}{2}M_1R^2\,\alpha\,.$$

Point of contact of the string with the disc is accelerating up with acceleration a and the center of the disk is accelerating down with the same acceleration a. Therefore, relative acceleration of the center of the disc with respect to the string is $2a$. Hence

$$\alpha = \frac{2a}{R} \quad\rightarrow\quad T = M_1a$$

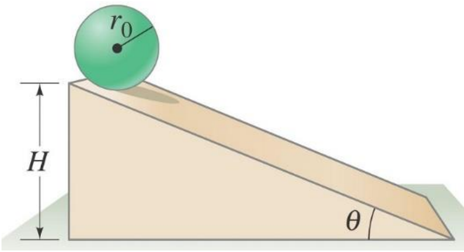
$$M_1g - T = M_1a \quad\rightarrow\quad a = \frac{g}{2}$$

$$M_2g - T = M_2a \rightarrow \quad M_2g - \frac{1}{2}M_1g = M_2\left(\frac{g}{2}\right) \quad\rightarrow\quad M_1 = M_2 \quad\rightarrow\quad \frac{M_1}{M_2} = 1$$

(c)

$$\alpha = \frac{2a}{R} \quad\rightarrow\quad \alpha = \frac{g}{R}.$$

2. A solid, uniform sphere of mass M and radius r_0 starts from rest and rolls without slipping down an inclined plane of height H, and angle of inclination θ.



(For the solid uniform sphere $I_{CM} = 2Mr_0^2/5$.)

(a) (6 Pts.) What is the translational speed of the sphere when it reaches the bottom of the inclined plane?

(b) (6 Pts.) What is its angular acceleration?

(c) (8 Pts.) What should be the minimum coefficient of static friction between the sphere and the inclined plane for the sphere to roll without slipping?

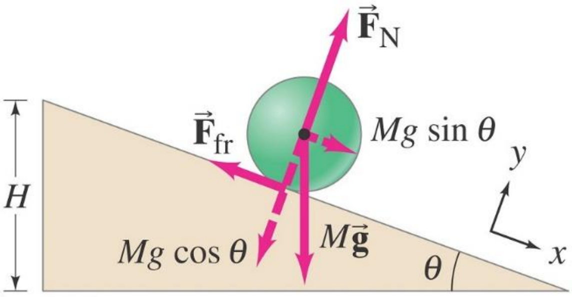
Solution:

(a) Total mechanical energy is conserved throughout the motion.

$$E_{\rm i} = MgH\,, \qquad E_{\rm f} = \frac{1}{2}Mv_{\rm c}^2 + \frac{1}{2}I\omega^2\,, \qquad \omega = \frac{v_{\rm c}}{r_0}$$

$$E_f = E_i \;\rightarrow\; \frac{1}{2}Mv_{\rm c}^2 + \frac{1}{2}\left(\frac{2}{5}Mr_0^2\right)\left(\frac{v_c}{r_0}\right)^2 = MgH \;\rightarrow\; v_c = \sqrt{\frac{10}{7}gH}\,.$$

(b)



$$Mg\sin\theta - F_{fr} = Ma\,, \qquad F_N - Mg\cos\theta = 0$$

$$r_0F_{fr} = \left(\frac{2}{5}Mr_0^2\right)\alpha = \left(\frac{2}{5}Mr_0^2\right)\frac{a}{r_0} \;\rightarrow\; F_{fr} = \frac{2}{5}Ma$$

$$Mg\sin\theta - \frac{2}{5}Ma = Ma \;\rightarrow\; a = \frac{5}{7}g\sin\theta$$

$$\alpha = \frac{a}{r_0} \;\rightarrow\; \alpha = \frac{5g}{7r_0}\sin\theta.$$

(c)

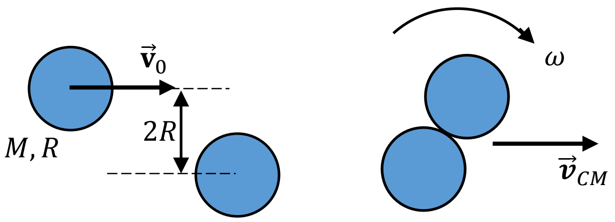
$$F_{fr} = \frac{2}{5}Ma = \frac{2}{7}Mg\sin\theta\,, \qquad F_N = Mg\cos\theta$$

$$F_{fr} \le \mu_s F_N \;\rightarrow\; \mu_s \ge \frac{F_{fr}}{F_N} \;\rightarrow\; \mu_s \ge \frac{2}{7}\tan\theta\,.$$

3. A uniform disk of mass M and radius R slides without rotating across a frictionless horizontal icy surface at speed v_0. It makes a glancing collision with an identical disk which is initially at rest such that their rims just touch. Because their rims are coated with instant acting glue, the pucks stick together following the collision. The moment of inertia of a uniform disk about its center is $I_C = MR^2/2$.

(a) (8 Pts.) Find the center of mass speed of the combined two-disc system after the collision.

(b) (12 Pts.) Find the angular speed of rotation around the center of mass after the collision.



Solution:

(a1) The center of mass speed before the collision is

$$V_{CM} = \frac{Mv_0}{2M} \rightarrow \quad V_{CM} = \frac{1}{2}v_0.$$

The speed of the center of mass does not change during the collision.

(a2) Linear momentum is conserved during the collision. Therefore,

$$Mv_0 = 2M\, V_{CM} \rightarrow \quad V_{CM} = \frac{1}{2}v_0$$

(b) Angular momentum about the center of mass is also conserved.

$$L_i = MRv_0\,, \qquad L_f = I\omega$$

Moment of inertia of the two disks about their common center of mass is

$$I = 2\left(\frac{1}{2}MR^2 + MR^2\right) \rightarrow \quad I = 3MR^2$$

Therefore,

$$L_f = L_i \rightarrow \quad \omega = \frac{v_0}{3R}$$

4. A space ship is stranded without any fuel in a circular orbit around a star. The radius of the orbit is R and the speed of the space ship is v_0.

(a) (6 Pts.) What is the mass of the star?

In order to escape from this obit and get back to our solar system the captain of the ship designs an explosion that will split the ship into two equal parts both with mass $m/2$. After the explosion one half of the ship will remain near the star, and the other half carrying the passengers will escape from the star system to our solar system (which can be assumed to be infinitely far away)

(b) (7 Pts.) What should be the minimum speed of the escaping part after the explosion in terms of the given quantities so that it can get infinitely far away from the star?

(c) (7 Pts.) What is minimum energy of the explosion so that the escaping part can reach the escape speed?

Solution:

(a)

$$F_G = G\frac{Mm}{R^2}\,, \qquad F_G = \frac{mv_0^2}{R} \quad \rightarrow \quad M = \frac{Rv_0^2}{G}\,.$$

(b) Total energy of the escaping part after the explosion is

$$E = \frac{1}{2}\left(\frac{m}{2}\right)v_{\mathrm{min}}^2 - \frac{GM}{R}\left(\frac{m}{2}\right)$$

Minimum energy at infinity would be zero. Therefore

$$\frac{1}{2}\left(\frac{m}{2}\right)v_{\mathrm{min}}^2 - \frac{GM}{R}\left(\frac{m}{2}\right) = 0 \quad \rightarrow \quad v_{\mathrm{min}}^2 = \frac{2GM}{R} = 2v_0^2 \quad \rightarrow \quad v_{\mathrm{min}} = \sqrt{2}v_0$$

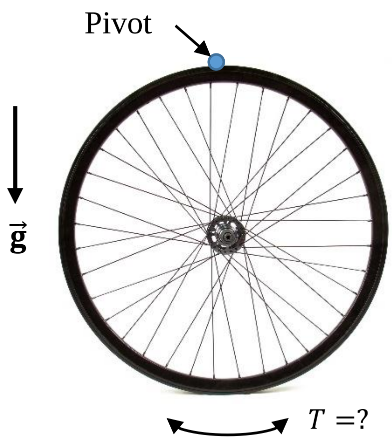
(c) Momentum is conserved during the explosion.

$$p_i = mv_0\,, \qquad p_f = \left(\frac{m}{2}\right)v_1 + \left(\frac{m}{2}\right)v_{\mathrm{min}} \quad \rightarrow \quad v_1 = \left(2-\sqrt{2}\right)v_0\,.$$

$$E_i = \frac{1}{2}mv_0^2\,, \qquad E_f = \frac{1}{2}\left(\frac{m}{2}\right)\left(2-\sqrt{2}\right)^2 v_0^2 + \frac{1}{2}\left(\frac{m}{2}\right)2v_0^2 = \frac{1}{2}\left(3-2\sqrt{2}\right)mv_0^2 + \frac{1}{2}mv_0^2$$

$$\Delta E = \left(3-2\sqrt{2}\right)\frac{1}{2}mv_0^2 = \left(3-2\sqrt{2}\right)E_i\,.$$

5. (20 Pts.) A bicycle wheel of mass M and radius R has a moment of inertia I_0 around its center of mass, which is at its center. If the wheel is suspended on a wall from its rim with a frictionless pivot as shown in the figure, find the period of its small oscillations. Gravitational acceleration is given as g.



Solution:

(1) Use Newton's second law.

$$\tau = I\alpha \quad \rightarrow \quad -Mg R \sin\theta = (I_0 + MR^2)\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{MgR}{I_0 + MR^2}\theta = 0 \quad \rightarrow \omega = \sqrt{\frac{MgR}{I_0 + MR^2}} \quad \rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_0 + MR^2}{MgR}}$$

Identify the system as a physical pendulum

$$\omega = \sqrt{\frac{MgR}{I_P}}$$

and use parallel axis theorem to note that $I_P = I_0 + MR^2$.